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On a Simple Wave Approximation to a Non-Linear Problem

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SUMMARY
The range of validity of a simple wave approximation to a non-linear set of two dissipative wave equations has been studied. The non-linear set is, when the dissipative terms are omitted, totally exceptional. It describes the propagation of longitudinal waves in an ideal elastic bar with some viscous stress. Upon a non-linear transformation, the equations become linear. These linear equations have been studied first. The results for the non-linear equations are then easily obtained by transforming backwards. It turns out that, if the non-linearity is small enough, they are similar to those obtained for the linear equations.

1. Introduction

1.1. *Statement of the Problem*

In physics one occasionally has to deal with sets of partial differential equations which are both non-linear and either dissipative or dispersive. An example of such a set might be

$$\alpha_t + [1+\varepsilon\phi(\alpha, \beta)]\alpha_x = \mu(\alpha_{xx} - \beta_{xx}), \tag{1}$$

$$\beta_t - [1+\varepsilon\psi(\alpha, \beta)]\beta_x = \mu(\beta_{xx} - \alpha_{xx}), \tag{2}$$

where x runs through the interval $(-\infty, \infty)$, t through $[0, \infty)$, $\phi(\alpha, \beta)$ and $\psi(\alpha, \beta)$ are continuous, often even monotonic functions of α and β, μ and ε are real positive constants usually much smaller than one and the subscripts x, t denote partial differentiation with respect to x, respectively t. A well known example of equations of this type is found in Lighthill's theory of waves in a real gas (Lighthill [1]).

An exact and complete solution for these equations is, at present, beyond all possibilities. Therefore, various approximations have to be used. In this paper we are concerned with a problem arising in an approximation method used by Lighthill. It applies to a certain class of initial value problems for (1) and (2), viz.:

$$\alpha(x, 0) = f(x), \tag{3}$$

$$\beta(x, 0) = \beta_0, \tag{4}$$

where β_0 is a constant which can be taken equal to zero without any loss of generality.

When $\mu=0$, it is easily seen that (2) is satisfied identically. Then, (1) becomes a first order equation in α, which is easily solved. The resultant solution is a simple wave solution for the hyperbolic set obtained by putting $\mu=0$.

Now, Lighthill's approximation, which for obvious reasons will be called the simple wave approximation henceforth, is based on the assumption that, when the initial conditions (3) and (4) are prescribed for the equations (1) and (2) with μ small but not zero, β will be negligible, at any rate for some finite interval of time. In this way one obtains from (1):

$$\alpha_t + [1+\varepsilon\phi(\alpha, 0)]\alpha_x = \mu\alpha_{xx}, \tag{5}$$

which is an equation of Burgers type. In Lighthill's example ϕ was linear in α. The exact solution of the initial value problem is known in that case. Now, the problem is that β will grow slowly from zero and therefore it is not at all obvious that α satisfies (5) for longer intervals of time too. In general to answer this question would present rather formidable difficulties. In the present

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paper these will be circumvented by choosing a special form for (1) and (2) for which the question can be answered. In this example it turns out that α and β may eventually become of the same order.

1.2. *Choice of the Example and Method of Solution*

Our problem could be stated in the following terms. When $\mu=0$ we have as solution a simple wave. That is a pure α-wave running to the right (towards positive x). The dissipative terms in the right-hand side provide some coupling. This coupling presumably will cause both the appearance of an α-wave running to the left and β-waves in both directions. A first complication is that, when $\mu=0$, a shock wave might develop. Undoubtedly this is the most general and physically the most important case. However, to keep things as simple as possible, we will avoid this which can be done by considering only so called totally exceptional equations in the sense of Lax [2]. In these equations ϕ depends on β only, ψ on α. Then, it is easily seen that the characteristic speed in a simple wave is constant, therefore no shock wave develops.

The second point is that it seems plausible to assume that the exact form of the right-hand side of (1) and (2) is not of great consequence for our problem as long as the leading terms are of the type indicated. This enables us to choose a form which can be transformed into a linear set of equations by means of a non-linear transformation. These linear equations can be solved formally. Application of the inverse transformation, then supplies the answer to our questions. As a matter of fact most of the information needed can be obtained from the solutions of the linear equations directly. In section 2 the set of non-linear equations having the required properties as well as the linear system obtained by using a non-linear transformation will be given. They admit a conceivable physical interpretation. The linear equations have been treated, in connection with the simple wave approximation, extensively in [3]. Some of the mathematical results obtained in that paper and some new ones will be discussed and interpreted in sections 3, 4, 5 and 6. In section 7 we return to the non-linear equations.

2. The model Equations

A suitable set of equations can be derived from the equations for longitudinal waves in an ideal elastic bar (Broer [4]) by adding a viscous stress term. The coefficient of viscosity is some function of the density. It is possible to choose this function in such a way that the equations become linear upon transformation to moving (Lagrangian) coordinates. We assume therefore the mass and momentum equations in the form

$$\rho_t + v\rho_x + \rho v_x = 0\,, \tag{1}$$

$$\rho v_t + \rho v v_x = Y_0(\rho^{-1})_x + (\mu\rho_0^2\rho^{-1}v_x)_x\,, \tag{2}$$

where ρ is the density, v the velocity, Y_0 a constant viz. Young's modulus. In an ideal elastic bar the specific energy of deformation is $\frac{1}{2}Y_0(\rho^{-1}-\rho_0^{-1})^2$, the stress $Y_0(\rho^{-1}-\rho_0^{-1})$. μ is a small positive constant of the dimension of a kinematic viscosity coefficient. The subscript zero refers to the unstrained situation. The sound speed a is given by

$$a^2 = Y_0\rho^{-2}\,. \tag{3}$$

Its value a_0 for $\rho=\rho_0$ will be useful as a reference speed. When $\mu=0$ the equations are hyperbolic and the characteristic variables are $\alpha=a-v$ and $\beta=a+v$.

It is easy to write (1) and (2) in terms of these variables. For our purposes it is convenient to make the equations dimensionless by putting:

$$x = Lx', \quad t = La_0^{-1}t', \quad \mu = 2a_0L\mu'\,,$$

$$\alpha = a_0[1+2\varepsilon\alpha'], \quad \beta = a_0[1+2\varepsilon\beta']\,.$$

In these equations L is some reference length connected with the initial value $\alpha(x,0)$, c.q. the dominant wavelength, ε a dimensionless measure for the strength of the wave that mostly will be chosen such that the absolute maximum of the sum of the solutions α' and β' is equal to or smaller than one.

Performing the indicated substitutions in (1), (2) and (3), and dropping the accents we obtain:

$$\alpha_t + [1+2\varepsilon\beta]\alpha_x = -\mu[1+\varepsilon(\alpha+\beta)]\{[1+\varepsilon(\alpha+\beta)](\beta-\alpha)_x\}_x, \tag{4}$$

$$\beta_t - [1+2\varepsilon\alpha]\beta_x = \mu[1+\varepsilon(\alpha+\beta)]\{[1+\varepsilon(\alpha+\beta)](\beta-\alpha)_x\}_x. \tag{5}$$

The equations are of the required form. When terms of $O(\varepsilon\mu)$ are dropped they reduce to special (and when $\mu=0$ totally exceptional) cases of (1.1) and (1.2). Now, we transform (4) and (5) to the Lagrangian coordinate $\rho_0 s = m$, where m is the mass coordinate as used in [4]. The details of this transformation will be stripped. We notice only the formulas

$$\left(\frac{\partial x}{\partial t}\right)_s = a_0^{-1} v = \varepsilon(\beta-\alpha), \tag{6}$$

$$\left(\frac{\partial x}{\partial s}\right)_t = \rho_0 \rho^{-1} = 1+\varepsilon(\alpha+\beta), \tag{7}$$

where α, β, x, s and t are dimensionless. For the transformed equations we find

$$\alpha_t + \alpha_s = \mu(\alpha_{ss} - \beta_{ss}), \tag{8}$$

$$\beta_t - \beta_s = \mu(\beta_{ss} - \alpha_{ss}), \tag{9}$$

which are linear indeed.

The initial conditions will be stated in the following way

$$\alpha(s, 0) = f(s), \tag{10}$$

$$\beta(s, 0) = 0. \tag{11}$$

3. Balance Equations

Some conservation laws and balance equations will be derived for (2.8) and (2.9). These equations themselves are in the form of a conservation law. Adding and subtracting them gives:

$$\frac{\partial}{\partial t}(\alpha+\beta) + \frac{\partial}{\partial s}(\alpha-\beta) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\alpha-\beta) + \frac{\partial}{\partial s}[\alpha+\beta-2\mu(\alpha_s-\beta_s)] = 0,$$

describing conservation of mass, respectively momentum.

For every natural number $n \geqq 2$, it is possible to construct two linearly independent balance equations of degree n. They may be written in the form:

$$\frac{\partial}{\partial t}\alpha^n + \frac{\partial}{\partial s}[\alpha^n + \mu n \alpha^{n-1}(\beta_s - \alpha_s)] + \mu(n-1)n\alpha^{(n-2)}\alpha_s(\alpha_s - \beta_s) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}\beta^n - \frac{\partial}{\partial s}[\beta^n + \mu n \beta^{n-1}(\beta_s - \alpha_s)] - \mu(n-1)n\beta^{(n-2)}\beta_s(\alpha_s - \beta_s) = 0, \tag{3}$$

and have been found from (2.8) and (2.9) by premultiplying the first one by α^{n-1} and the second one by β^{n-1}. When (2) and (3) are added and n has been put equal to 2 the equation of balance of energy (kinetic- + deformation energy) is found:

$$\frac{\partial}{\partial t}(\alpha^2+\beta^2) + \frac{\partial}{\partial s}[\alpha^2 - \beta^2 + 2\mu(\alpha-\beta)(\beta_s-\alpha_s)] + 2\mu(\alpha_s-\beta_s)^2 = 0. \tag{4}$$

From subtracting and putting $n=2$ a Bernoulli-like equation (when $\mu=0$ it is the exact Bernoulli-equation)

$$\frac{\partial}{\partial t}(\alpha^2-\beta^2)+\frac{\partial}{\partial s}[\alpha^2+\beta^2+2\mu(\alpha+\beta)(\beta_s-\alpha_s)]+2\mu(\alpha_s^2-\beta_s^2)=0$$

is found.

It is obvious that $\int_{-\infty}^{\infty}\alpha^2\,ds$, if it exists, may be seen as the total energy of the α-mode at time t. $\int_{-\infty}^{\infty}\beta^2\,ds$ can be given a similar interpretation. This will be used later on in the paper.

4. Some Mathematical and Physical Aspects of the Linear Equations

4.1. *Some Notations*

R: the interval $(-\infty,\infty)$ of the real numbers.
Q: a strip in the s–t plane containing all the points satisfying the inequalities $-\infty<s<\infty$ and $0<t<T<\infty$.

Consider scalar valued functions $u(s,t)$ defined on R (t fixed) and Q respectively.

$L_2(R)$ is a Hilbert-space containing all square integrable functions on R with inner product $(\,,)$ and norm $\|\ \|$ defined by

$$(u,v)=\int_{-\infty}^{\infty}u^*(s)v(s)ds\ ;\quad \|u\|=(u,u)^{\frac{1}{2}},$$

u^* being the complex conjugate of u.

The Sobolev-space $W_2^m(R)$ (m a natural number) is a Hilbert-space containing all $L_2(R)$ functions $u(s)$ that have generalised derivatives $D^k u\in L_2(R)$, where $k=1,\ldots,m$ (Smirnow [5]). The inner product $(,)_m$ and norm $\|\ \|_m$ are respectively

$$(u,v)_m=\sum_{i=0}^{m}(D^i u,D^i v);\quad \|u\|_m=(u,u)_m^{\frac{1}{2}}.$$

$L_2^\Delta(R)$ is a Hilbert-space containing all functions $u\in L_2(R)$, of which the Fourier transform $\bar{u}(k)$ defined by

$$\bar{u}(k)=\int_{-\infty}^{\infty}u(s)\exp(-iks)ds \tag{1}$$

vanishes identically outside a finite interval $[-\Delta,\Delta]$ ($\Delta\in R$), with inner product $(\,,)_{R,\Delta}$ and norm $\|\ \|_{R,\Delta}$ defined by

$$(u,v)_{R,\Delta}=\int_{-\infty}^{\infty}u^*(s)v(s)ds,\quad \|u\|_{R,\Delta}=(u,u)_{R,\Delta}^{\frac{1}{2}}.$$

Where not stated otherwise all integrations are in the sense of Lebesque and all differentials are meant in the generalised sense, although the classical notation will be retained. The Fourier transform with respect to s of a function $u(s,t)$ will sometimes be called the spectrum of u.

4.2. *Existence and Uniqueness*

In [3] it has been proved that equations (2.8) and (2.9) are uniquely solvable for every $f\in L_2(R)$ ($W_2^m(R)$, $L_2^\Delta(R)$) and that for every $0<t<T<\infty$ the solution is an element of $L_2(R)$ ($W_2^m(R)$, $L_2^\Delta(R)$). Furthermore $\alpha\to f$ and $\beta\to 0$ as $t\to 0$ in the sense of the $L_2(R)$ ($L_2^\Delta(R)$) norm. The solutions may be represented by

$$\alpha(s,t)=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}1+\int_{-\infty}^{\infty}2\right]g^{(2)}(k)\exp[h(k,\xi)t]dk, \tag{2}$$

$$\beta(s,t)=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}1+\int_{-\infty}^{\infty}2\right]g^{(1)}(k)\exp[h(k,\xi)t]dk, \tag{3}$$

where

$$g^{(1)}(k) = -\tfrac{1}{2}i\mu k(1-\mu^2k^2)^{-\frac{1}{2}}\bar{f}(k)\,, \quad g^{(2)}(k) = \tfrac{1}{2}[-1+(1-\mu^2k^2)^{\frac{1}{2}}](1-\mu^2k^2)^{-\frac{1}{2}}\bar{f}(k)\,,$$

$$h(k,\xi) = ik(1-\mu^2k^2)^{\frac{1}{2}} - \mu k^2 + ik\xi\,,$$

$$\xi = st^{-1}$$

and $\bar{f}(k)$ is defined similar to (1).

The number 1 respectively 2 through the integration symbol means integration in the first-, respectively second sheet of the complex k-plane. The first sheet is defined by

$$\lim_{|k|\to\infty} \frac{(1-\mu^2k^2)^{\frac{1}{2}}}{\mu k} = -i \qquad (0 \leqq \arg k \leqq \pi)\,,$$

and the second by

$$\lim_{|k|\to\infty} \frac{(1-\mu^2k^2)^{\frac{1}{2}}}{\mu k} = i \qquad (0 \leqq \arg k \leqq \pi)\,.$$

This corresponds to cutting the k-plane from $-\infty$ to $-\mu^{-1}$ and from μ^{-1} to ∞.

4.3. *Stability and Positive Definiteness of* ρ.

It is shown in [3] that for every $f \in L_2(R)$ the solution of (2.8), ..., (2.11) is stable in the sense that for all $t \geqq 0$

$$\|\alpha(t)\|^2 + \|\beta(t)\|^2 \leqq \|f\|^2\,,$$

which means that the total energy of the system is bounded from above by the initial energy. This result has been derived using the equation of balance of energy (3.4). In a physical problem ρ, the density, must be essentially positive i.e. $\rho \geqq \delta > 0$. Now, one may wonder whether it is possible to choose the initial conditions and ε in such a way that this is satisfied. Let $f \in W_2^1(R)$ and $\|f\|_1 \leqq \sqrt{2}$. From [3] we infer the existence of $\tilde{\alpha}(s,t)$ and $\tilde{\beta}(s,t)$, continuously depending on s, such that for all $t \geqq 0$

$$\alpha(s,t) = \tilde{\alpha}(s,t) \quad \text{(a.e.)}\,, \qquad \beta(s,t) = \tilde{\beta}(s,t) \quad \text{(a.e.)}\,,$$

$$\sup_{s\in R} |\tilde{\alpha}| + |\tilde{\beta}| \leqq 1\,,$$

the last condition being equivalent to the one posed in section 2 concerning the definition of ε. It is clear that for $\varepsilon \leqq 1-\delta' < 1$, ρ is positive indeed.

5. Monochromatic Waves

Following section 1 the simple wave approximation of the set of linear equations under consideration is given by

$$\alpha_{0t} + \alpha_{0s} - \mu\alpha_{0ss} = 0\,, \tag{1}$$

$$\alpha_0(s,0) = f(s)\,. \tag{2}$$

In this section we shall deal with the particular simple case of solutions $(\alpha, \beta, \alpha_0)$ that are periodic functions with respect to s. This may serve as an introduction to the more difficult problems arising in dealing with a general initial function $f(s)$.

Let

$$f(s) = \exp(ik_1 s) \quad (k_1 \text{ real})\,.$$

The solutions α and β may formally be found from (4.2) and (4.3) by substituting $\bar{f}(k) = 2\pi\delta(k-k_1)$, where $\delta(x)$ is the Dirac δ-function. We find for $|k| \leqq \mu^{-1}$

$$\alpha = \frac{1+c}{2c} \exp(iks - ikct - \mu k^2 t) + \frac{c-1}{2c} \exp(iks + ikct - \mu k^2 t) , \tag{3}$$

$$\beta = \frac{i\mu k}{2c} \exp(iks - ikct - \mu k^2 t) - \frac{i\mu k}{2c} \exp(iks + ikct - \mu k^2 t) \tag{4}$$

and for $|k| \geqq \mu^{-1}$

$$\alpha = \frac{C-i}{2C} \exp(iks + kCt - \mu k^2 t) + \frac{C+i}{2C} \exp(iks - kCt - \mu k^2 t) , \tag{5}$$

$$\beta = \frac{\mu k}{2C} \exp(iks + kCt - \mu k^2 t) - \frac{\mu k}{2C} \exp(iks - kCt - \mu k^2 t) , \tag{6}$$

where

$$c(k) = |(1-\mu^2 k^2)^{\frac{1}{2}}| \qquad (|k| \leqq \mu^{-1}) ,$$
$$C(k) = |(\mu^2 k^2 - 1)^{\frac{1}{2}}| \qquad (|k| \geqq \mu^{-1})$$

and, for convenience, the subscript 1 has been dropped again.

α_0 is given by

$$\alpha_0 = \exp(iks - ikt - \mu k^2 t) . \tag{7}$$

(3) and (4) clearly demonstrate the development of right- and left moving waves. $c(k)$ can be seen as a phase velocity. When $|k| \geqq \mu^{-1}$, we can speak of travelling waves no longer. Substituting $\sin(kct) = [\exp(ikct) - \exp(-ikct)]/2i$ in (3) and (4), we find for $|k| \leqq \mu^{-1}$

$$\alpha = \left[\exp(-ikct) + \frac{i(c-1)}{c} \sin(kct) \right] \exp(iks - \mu k^2 t) , \tag{8}$$

$$\beta = \frac{\mu k}{c} \sin(kct) \exp(iks - \mu k^2 t) ,$$

showing that α may also be seen as a superposition of a right moving- and a standing-, β as a pure standing wave.

If $|\mu k| < 1$ we expand $(1-\mu^2 k^2)^{\frac{1}{2}}$ around $\mu k = 0$. In this way we find from (8)

$$\alpha = [1 + \tfrac{1}{2} i\mu^2 k^3 t + \ldots] \exp(iks - ikt - \mu k^2 t) + [-\tfrac{1}{2} i\mu^2 k^2 \sin(kt) + \ldots] \exp(iks - \mu k^2 t) . \tag{9}$$

However, as α is an analytic function of μk for all finite k (see (3)), this expansion also holds for $|\mu k| \geqq 1$ and so, for all finite k.

We have, using (7) and (8)

$$\alpha - \alpha_0 = \tfrac{1}{2} i\mu^2 k^3 t \exp(iks - ikt - \mu^2 k^2 t) - \tfrac{1}{2} i\mu^2 k^2 \sin(kt) \exp(iks - \mu k^2 t) + \ldots , \tag{10}$$

from which we infer that the difference between α and α_0 is "small" if $\mu^2 |k|^3 t \ll 1$.

The expansion (9) may also be found in a different way which will turn out to be succesful for $f \in L_2^4(R)$ too.

Write

$$\alpha = \bar{\alpha}(k, t) \exp(iks) , \qquad \beta = \bar{\beta}(k, t) \exp(iks) ,$$

then $\bar{\alpha}$ and $\bar{\beta}$ satisfy

$$\bar{\alpha}_t + (ik + \mu k^2)\bar{\alpha} = \mu k^2 \bar{\beta} , \qquad \bar{\beta}_t + (-ik + \mu k^2)\bar{\beta} = \mu k^2 \bar{\alpha} ,$$

and so

$$\bar{\alpha}_{tt} + 2\mu k^2 \bar{\alpha}_t + (\mu^2 k^4 + k^2)\bar{\alpha} = \mu^2 k^4 \bar{\alpha} . \tag{11}$$

The initial data for (11) become

$$\bar{\alpha}(k, 0) = 1 , \qquad \bar{\alpha}_t(k, 0) = -ik - \mu k^2 .$$

Now, it is quite elementary to show that $\bar{\alpha}$ satisfies the integral equation

$$\bar{\alpha}(k,t) = \bar{\alpha}_0(k,t) + \mu^2 k^3 \int_0^t \sin[k(t-\tau)] \exp[-\mu k^2(t-\tau)]\,\bar{\alpha}(k,\tau)\,d\tau\,,$$

where $\bar{\alpha}_0(k,t) = \alpha_0(s,t)\exp(-iks)$.

The solution of this equation may be found by means of iteration:

$$\bar{\alpha}^{(0)}(k,t) = \bar{\alpha}_0(k,t)\,,$$

$$\bar{\alpha}^{(n)}(k,t) = \mu^2 k^3 \int_0^t \bar{\alpha}^{(n-1)}(k,\tau) \sin[k(t-\tau)] \exp[-\mu k^2(t-\tau)]\,d\tau \qquad (n=0,1,\ldots)\,,$$

so $\alpha = \Sigma_{n=0}^{\infty} \bar{\alpha}^{(n)}(k,t)\exp(iks)$ (for a proof, see [3]).

Some computations show this expansion to be identical to the one found before.

For β a similar procedure may be followed.

From (10) we see that as $t \to \infty$ the simple wave approximation breaks down. This turns out not to be true when $f \in L_2^{\Delta}(R)$, as will be shown in the next section. Looking at (3), ..., (6) we observe that as $t \to \infty$ the dissipation at high frequencies k is much larger than at low frequencies. When $f \in L_2^{\Delta}(R)$ the solution α (see (4.2)) may be seen as a superposition (integral with respect to k) of monochromatic solutions. When $t \to \infty$, only the values in a small $\{O(t^{-\frac{1}{2}})$ as was proved in [3]$\}$ vicinity of $k=0$ will contribute significantly to the integral. This "explains" why $f \in L_2^{\Delta}(R)$, as $t \to \infty$, leads to results different from those found for monochromatic waves. In particular it will turn out that, as $t \to \infty$, the simple wave approximation holds again.

6. $L_2^{\Delta}(R)$ Solutions

6.1 *An Expansion of the Solution*

To get some insight in the problem stated by (1.1), ..., (1.4), one sometimes uses an expansion in a series of α and β, where the solution of the simple wave approximation (1.3) and (1.5) is used as the first term in the expansion of α (cf. Lighthill [1]).

The convergence of such an expansion, as far as we know, never has been treated. In general this would be very complicated. However, it has turned out to be possible to show convergence of such an expansion for the simple case treated here.

Let $f \in L_2^{\Delta}(R)$ and $\alpha^{(2n)}$ and $\beta^{(2n-1)}$ satisfy

$$\alpha_t^{(0)} + \alpha_s^{(0)} - \mu\alpha_{ss}^{(0)} = 0, \quad \alpha_t^{(2n)} + \alpha_s^{(2n)} - \mu\alpha_{ss}^{(2n)} = -\beta_{ss}^{(2n-1)}\,,$$

$$\beta_t^{(2n-1)} - \beta_s^{(2n-1)} - \mu\beta_{ss}^{(2n-1)} = -\alpha_{ss}^{(2n-2)} \qquad (n = 1, 2, \ldots)$$

and let $\alpha^{(0)}(s,0) = f(s)$, $\alpha^{(2n)}(s,0) = \beta^{(2n-1)}(s,0) = 0$.

In [3] it has been shown that for all finite $t \geqq 0$, $\Sigma_{n=0}^{N} \alpha^{(2n)}\mu^{2n}$ converges to α, $\Sigma_{n=0}^{N} \beta^{(2n+1)}\mu^{2n+1}$ converges to β as $N \to \infty$ in the sense of the $L_2^{\Delta}(R)$ norm. The method used there runs along lines quite similar to those used in section 5 to obtain an expansion in a series of monochromatic solutions.

Other important results, found in [3], are given by

$$\left\| \alpha - \sum_{n=0}^{N} \mu^{2n}\alpha^{(2n)} \right\| \leqq \left(\sum_{n=N+1}^{\infty} t^n \mu^{2n} \Delta^{3n} \right) \|\alpha\|_{R,\Delta}\,, \tag{1}$$

$$\left\| \beta - \sum_{n=0}^{N} \mu^{2n+1}\beta^{(2n+1)} \right\|_{R,\Delta} \leqq \mu\Delta \left(\sum_{n=N+1}^{\infty} t^n \mu^{2n} \Delta^{3n} \right) \|\alpha\|_{R,\Delta}\,,$$

and will be used repeatedly in the next sections. Finally we add the remark that it is possible to prove convergence for functions of which the spectrum is not of bounded support. However these functions will not be treated here.

6.2. *The Start of the β-Mode and the Left Running α-Mode*

From our considerations in the sections 1.2 and 5 we expect the appearance of a left running α-wave and β-waves in both directions. Let $\Delta^3\mu^2 T \ll 1$, then for every $t\in[0, T]$:

$$\|\alpha-\alpha_0-\mu^2\alpha^{(2)}\|_{R,\Delta} \ll \|\alpha\|_{R,\Delta}\,,$$

$$\|\beta-\mu\beta^{(1)}\|_{R,\Delta} \ll \|\alpha\|_{R,\Delta}\,,$$

which implies that for every $t\in[0, T]$, $\alpha_0+\mu^2\alpha^{(2)}$ is a good approximation to α and so is $\mu\beta^{(1)}$ to β. As is easily seen

$$\alpha_0+\mu^2\alpha^{(2)} = \tfrac{1}{8}\mu^2(\pi\mu t)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{d^2 f}{d\xi^2}\exp\left[-\frac{(s-\xi+t)^2}{4\mu t}\right]d\xi +$$

$$+\tfrac{1}{2}(\pi\mu t)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\left[f(\xi)-\tfrac{1}{2}\mu^2 t\frac{d^3 f}{d\xi^3}-\tfrac{1}{4}\mu^2\frac{d^2 f}{d\xi^2}\right]\exp\left[-\frac{(s-\xi-t)^2}{4\mu t}\right]d\xi\,,$$

$$\mu\beta^{(1)} = \frac{\mu}{4}(\pi\mu t)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{df}{d\xi}\left\{\exp\left[-\frac{(s-\xi-t)^2}{4\mu t}\right]-\exp\left[-\frac{(s-\xi+t)^2}{4\mu t}\right]\right\}d\xi\,,$$

which confirms our expectations. Some insight in the initial state of the α- and β-mode may be gained by using an asymptotic expansion as $t\to 0$ (appendix 1). We find

$$\alpha_0+\mu^2\alpha^{(2)} = f(s-t)+\frac{\mu^2}{4}\left[\frac{d^2 f}{ds^2}(s+t)-\frac{d^2 f}{ds^2}(s-t)\right]-\frac{\mu^2 t}{2}\frac{d^3 f}{ds^3}(s-t)+O(t)\,,$$

$$\beta^{(1)} = \frac{1}{2}\left[\frac{df}{ds}(s-t)-\frac{df}{ds}(s+t)\right]+O(t)\,.$$

6.3. *The Simple Wave Approximation*

For solutions α which are square integrable α_0 will be called a useful (good-) approximation to α in the interval of time $[t_1, t_2]$ $(t_2>t_1)$ if for every $t\in[t_1, t_2]$

$$\|\alpha-\alpha_0\| \ll \|\alpha\|\,. \tag{2}$$

Of course, if (2) is satisfied α_0 locally still may deviate considerably from α.

Using (2) we immediately find

$$\|\alpha-\alpha_0\|_{R,\Delta} \leqq (e^{\Delta^3\mu^2 t}-1)\,\|\alpha\|_{R,\Delta}$$

and so, for every $t\in[0, T]$, where $\Delta^3\mu^2 T\ll 1$, α_0 is a good approximation in the sense of (2). This result is entirely similar to the one found in section 5 where we dealt with monochromatic solutions.

In [3] it has been shown that the simple wave approximation may fail for some finite time, but as $t\to\infty$ it holds again as then a positive constant K exists such that

$$\|\alpha-\alpha_0\|_{R,\Delta} \leqq K t^{-\frac{1}{2}}\|\alpha\|_{R,\Delta}\,.$$

This result has already been discussed in section 5. In this context we may notice the following interesting relation, used in some proofs in [3]:

$$\|\alpha\|^2 = \|\beta\|^2+\|\alpha_0\|^2\,.$$

It holds for every $f\in L_2(R)$.

7. The Non-Linear Equations

7.1. *The Inverse Transformation*

Let $f \in W_2^3(R)$ and absolutely integrable on R. Then, α and β are elements of $W_2^3(R)$ too and according to (3.1) $\int_{-\infty}^{s} (\tilde{\alpha}+\tilde{\beta})ds'$ exists in the sense of Riemann. So, by integrating (2.6) and (2.7) we find for every finite $t \geqq 0$:

$$s = x - \varepsilon \int_{-\infty}^{s} [\tilde{\alpha}(s', t) + \tilde{\beta}(s', t)]\, ds' . \tag{1}$$

We are interested in the conditions to be satisfied by $f(s)$ and ε that are sufficient for s to be solvable from (1) as an univalent function of x and t. Let $\| f \|_1 \leqq \sqrt{2}$, $\varepsilon \leqq 1-\delta < 1$ again. Define

$$s_0 = x , \tag{2}$$

$$\vdots$$

$$s_n = x - \varepsilon \int_{-\infty}^{s_{n-1}} [\tilde{\alpha}(s', t) + \tilde{\beta}(s', t)]\, ds' . \tag{3}$$

From section 4.3 we deduce $\sup_{s\in R} |\tilde{\alpha}+\tilde{\beta}| \leqq 1$, so $(\partial x/\partial s)_t$ is essentially positive and consequently (1) is uniquely solvable. Furthermore

$$|s_{n+1} - s_n| = \varepsilon \left| \int_{s_{n-1}}^{s_n} (\tilde{\alpha}+\tilde{\beta})\, ds' \right| \leqq (1-\delta)|s_n - s_{n-1}| ,$$

which implies that the sequence defined in (2) and (3) converges to $s(x, t)$ for every finite t.

It is easily verified that

$$s = \sum_{n=0}^{\infty} \varepsilon^n s^{(n)} ,$$

where

$$s^{(0)} = x ,$$

$$s^{(1)} = \int_{-\infty}^{x} [\tilde{\alpha}(s', t) + \tilde{\beta}(s', t)]\, ds' ,$$

$$\vdots$$

$$s^{(n)} = -\varepsilon^{1-n} \int_{\sum_{j=0}^{n-2} \varepsilon^j s^{(j)}}^{\sum_{j=0}^{n-1} \varepsilon^j s^{(j)}} [\tilde{\alpha}(s', t) + \tilde{\beta}(s', t)]\, ds' \qquad (n = 2, 3, \ldots) .$$

7.2. *On a Simple Wave Approximation*

Consider (5.1) and (5.2) where s is replaced by x as the simple wave approximation to the non-linear problem. This is not entirely equivalent to section 1, as the initial value $\alpha_0(x, 0)$ should have been equal to $f[s(x, 0)]$. However, this is just a mathematical difference and is not essential to the problem as all the aspects of approximating a non-linear problem by a linear one are retained.

Besides we will choose ε and $f(s)$ such that at $t=0$ the simple wave approximation does hold indeed.

Let $f \in L_2^4(R)$ and absolutely integrable such that (a) $\| f \|_1 \leqq \sqrt{2}$, (b) $\int_{-\infty}^{\infty} |f(s)|\, ds = M$ (a positive constant), (c) $\bar{f}(k)$ is analytic in a vicinity of $k=0$ and let $\varepsilon \leqq 1-\delta < 1$. Define:

$$\phi(s, t) = \int_{-\infty}^{s} [\tilde{\alpha}(s', t) + \tilde{\beta}(s', t)]\, ds' .$$

Using $|\tilde{\alpha}+\tilde{\beta}| \leqq 1$, Schwarz's inequality, change of order of integration and $\int_{\Delta}^{\Delta} k^2 |\bar{g}(k)|^2 dk \leqq \Delta^2 \int_{\Delta}^{\Delta} |\bar{g}(k)|^2 dk$, we find:

$$\int_{-\infty}^{\infty} |\alpha(x,t)-\alpha_0(x,t)|^2 dx =$$

$$= \int_{-\infty}^{\infty} |\alpha(s,t)-\alpha_0(s,t)+\alpha_0(s,t)-\alpha_0(s+\varepsilon\phi,t)|^2 [1+\varepsilon(\alpha+\beta)] ds$$

$$\leqq 2\int_{-\infty}^{\infty} |\alpha-\alpha_0|^2 ds + 2\int_{-\infty}^{\infty} |\alpha_0(s+\varepsilon\phi,t)-\alpha_0(s,t)|^2 ds \leqq$$

$$\leqq 2\int_{-\infty}^{\infty} |\alpha-\alpha_0|^2 ds + 2\varepsilon[\max_{s\in R}|\phi(s,t)|] \int_{-\infty}^{\infty} ds \int_{s}^{s+\varepsilon \max_{s\in R}|\phi|} \left|\frac{\partial\alpha_0(\sigma,t)}{\partial\sigma}\right|^2 d\sigma \leqq$$

$$\leqq 2\int_{-\infty}^{\infty} |\alpha-\alpha_0|^2 ds + 2\varepsilon^2[\max_{s\in R}|\phi(s,t)|^2] \int_{-\infty}^{\infty} \left|\frac{\partial\alpha_0(s',t)}{\partial s'}\right|^2 ds' \leqq$$

$$\leqq 2\int_{-\infty}^{\infty} |\alpha-\alpha_0|^2 ds + 2\varepsilon^2 \Delta^2 \{\max_{s\in R}|\phi(s,t)|\}^2 \int_{-\infty}^{\infty} |\alpha_0(s',t)|^2 ds', \tag{4}$$

where $0 \leqq \theta \leqq 1$.

Using the conservation law of mass (3.1) we find

$$\phi(s,t) = \int_{-\infty}^{s} f(s')ds' + \int_{0}^{t} [\tilde\beta(s,t')-\tilde\alpha(s,t')]dt',$$

which implies that

$$|\phi(s,t)| \leqq M+t. \tag{5}$$

Substituting this in (4) and using (6.1), we obtain that for all $t\in[0,T]$ (T finite)

$$\int_{-\infty}^{\infty} |\alpha-\alpha_0|^2 dx \leqq \{2[\exp(T\mu^2\Delta^3)-1]^2 + 2\varepsilon^2\Delta^2(M+T)^2\} \int_{-\infty}^{\infty} |\alpha_0|^2 dx,$$

which implies that if $M\varepsilon\Delta \ll 1$, $T \ll \Delta(\mu^4\Delta^4+\varepsilon^2)^{\frac{1}{2}}$ the simple wave approximation does hold indeed.

Of course we are also interested in the situation as $t\to\infty$. Now, a difficulty shows up as an inequality of the form (5) can be used no longer. However, in appendix 2 it has been shown that, as $t\to\infty$, $|\phi(s,t)| \leqq 4M$. This implies that given the condition $M\Delta\varepsilon \ll 1$ the simple wave approximation holds again. Therefore, when ε and/or Δ are chosen small enough the situation is entirely equivalent to the case treated in the former sections.

Appendices

Appendix 1

Define

$$K_{\pm}(s,t) = (\mu t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} g(x) \exp\left[-\frac{(x-s\pm t)^2}{4\mu t}\right] dx.$$

Lemma

Let $g\in L_2^A(R)$. As $t\to 0$, $K_{\pm}(s,t)$ has the following asymptotic expansion

$$K_{\pm}(s,t) \approx 2\sum_{n=0}^{\infty} (4\mu t)^n [(2n)!]^{-1} \Gamma(n+\tfrac{1}{2}) \frac{d^{2n}}{dx^{2n}} g(s\mp t), \tag{1}$$

where

$$\tilde{g}(x) = g(x) \text{ (a.e.)}$$

and $\tilde{g}(x)$ *is analytic on the real axis.*

Proof.

Define

$$\tilde{g}(x) = \frac{1}{2\pi} \int_{-\Delta}^{\Delta} \bar{g}(k) \exp(ikx) dk ,$$

then $g(x) = \tilde{g}(x)$ (a.e.) and $\tilde{g}(x)$ is analytic on the real axis. So $\tilde{g}(x)$ may substitute $g(x)$ in (1) and a positive number ρ exists such that for $|x - s \pm t| \leqq \rho$

$$\tilde{g}(x) = \sum_{n=0}^{\infty} a_n (x - s \pm t)^n , a_n = \frac{1}{n!} \frac{d^n}{dx^n} \tilde{g}(s \mp t) .$$

Furthermore, using formula 7.1.13 of Abramowitz and Stegun [6]:

$$\left| \left[\int_{-\infty}^{-\rho} + \int_{\rho}^{\infty} \right] (\mu t)^{-\frac{1}{2}} \tilde{g}(z + s \mp t) \exp[-z^2 (4\mu t)^{-1}] dz \right| \leqq$$

$$\leqq 8\rho^{-1} (\mu t)^{\frac{1}{2}} \left[1 + \left(1 + \frac{16\mu t}{\pi \rho^2} \right)^{\frac{1}{2}} \right]^{-1} \exp[-\rho^2 (4\mu t)^{-1}] \; (\max_{z \in R} |\tilde{g}(z)|) .$$

Now, all conditions required by de Bruijn [7], page 68, to construct an asymptotic expansion of $K_{\pm}(s, t)$ are satisfied. We find (1).

Appendix 2.

Lemma

Let $f(s)$ *satisfy the conditions required and* ϕ *be defined as in section* 7.2. *Then, as* $t \to \infty$

$$|\phi(s, t)| \leqq 4M .$$

Proof.

As is seen from (4.2) and (4.3)

$$I \overset{\text{def}}{=} \int_{-N}^{s} [\tilde{\alpha}(s', t) + \tilde{\beta}(s', t)] ds' =$$

$$= \left[\int_{-\Delta}^{\Delta} 1 + \int_{-\Delta}^{\Delta} 2 \right] (ik)^{-1} \psi(k) \{ \exp[h(k, st^{-1})t] - \exp[h(k, -Nt^{-1})t] \} dk , \tag{2}$$

where $N \gg 1$ and

$$\psi(k) = (4\pi)^{-1} (1 - \mu^2 k^2)^{-\frac{1}{2}} [1 - i\mu k + (1 - \mu^2 k^2)^{\frac{1}{2}}] \bar{f}(k) .$$

$\bar{f}(k)$ is analytic in a ρ-vicinity of $k = 0$ so we are able to choose a number $0 < \delta < \rho$ such that along $c_{\delta\pm} := \{k| \; |k| = \delta, \; -\frac{1}{2}\pi \pm \frac{1}{2}\pi \leqq \arg k \leqq \frac{1}{2}\pi \pm \frac{1}{2}\pi\}$

$$|\psi(k)| \leqq \tfrac{3}{4}\pi^{-1} |\bar{f}(0)| . \tag{3}$$

(2) may be rewritten in the form

$$I = \left[\int_{-\Delta}^{-\delta} (1+2) + \int_{\delta}^{\Delta} (1+2) + \int_{c_{\delta+}} (1+2) \right] (ik)^{-1} \psi(k) \{ \exp[h(k, st^{-1})t] - \exp[h(k, -Nt^{-1})t] \} dk . \tag{4}$$

As for all $k \in R$, where $|k| \geqq \delta$, Re $h(k, st^{-1}) \leqq -\mu\delta^2$, the first two integrals in the right-hand side of (4) are $O(t \exp(-\delta^2 \mu t))$ as $t \to \infty$. $(ik)^{-1} \psi(k)$ is analytic in a vicinity of $k=0$ in the first sheet of the complex k-plane. Using the method of saddle-points it is quite standard to derive

$$\left[\int_{c_{\delta+}} 1\right] (ik)^{-1} \psi(k) \{\exp[h(k, st^{-1})t] - \exp[h(k, -Nt^{-1})t]\} dk = O(t^{-\frac{1}{2}}) \qquad (t \to \infty).$$

Consider

$$I_1 = \left[\int_{c_{\delta+}} 2\right] (ik)^{-1} \psi(k) \exp[h(k, st^{-1})t] dk. \tag{5}$$

Choose $\delta < \frac{1}{2}\rho\mu^2$ and define $\varepsilon = 2\delta\mu^{-1}$. Let $s \geqq (1+\varepsilon)t$. Then by choosing δ small enough, Re $h \leqq 0$ along $c_{\delta+}$. Thus, substituting $k = \delta e^{i\phi}$ in (5) and using (3) we find $|I_1| \leqq \frac{3}{4}|\bar{f}(0)|$. If $s \leqq (1-\varepsilon)t$, then, to obtain Re $h \leqq 0$, we must shift the path of integration to $c_{\delta-}$. This leads to $|I_1| \leqq \frac{7}{4}|\bar{f}(0)|$. Now, let $(1-\varepsilon)t \leqq s \leqq (1+\varepsilon)t$. The saddle-point of $h(k, st^{-1})$ is located on the imaginary axis of the k-plane, inside the circle $|k| = \rho$ (cf. [3]). By using the method of saddle-points as developed by van der Waerden [8], we find that, as $t \to \infty$, $|I_1| \leqq \frac{5}{4}|\bar{f}(0)|$. Applying the results concerning I_1 to (4) and using the results obtained earlier in the proof we deduce the lemma.

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